TETRATION IN CONTEXT

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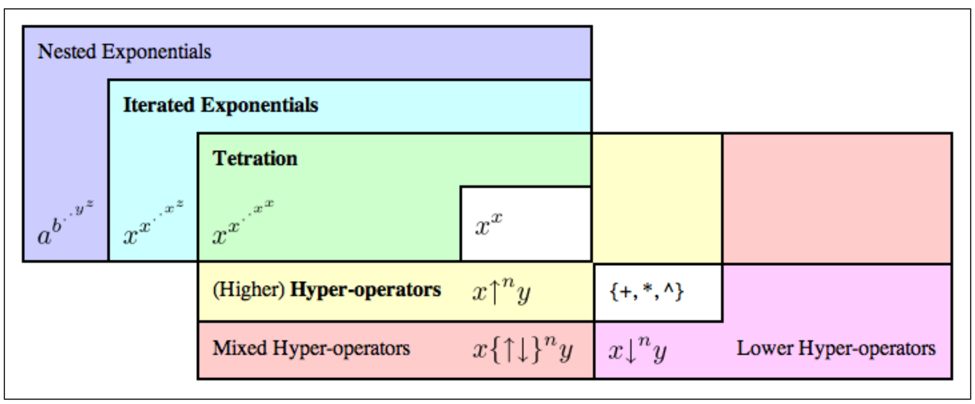
1. **Introduction**

The focus of this talk is on **tetration**, but during the course of discussion we will hit upon several related subjects and topics related to tetration. There are two major topics which are most closely related to tetration, which are commonly known as **iterated exponentials** and **hyper-operators**. Hyper-operators form a sequence generalizing the basic operators used throughout mathematics: addition, multiplication (iterated addition), exponentiation (iterated multiplication), tetration (iterated exponentials), pentation (iterated tetrationals), and so on.

2. **Terminology**

2.1. **Overview.** This figure shows several common terms related to tetration, and some examples of what kind of expressions that each process would involve.



2.2. **Univariate Terms for Bivariate Functions.** Iteration is a bivariate function. Iteration only works for **self-maps**, functions whose domain and codomain are the same set. A simple example of a self-map is a function $f : \mathbb{R} \to \mathbb{R}$. The iteration $f^n(x) = f^{n-1}(f(x)) = f(f^{n-1}(x))$ has two parameters: x is known as the state (or phase) parameter, and n is known as the time (or iteration) parameter. So there are two obvious ways of making this a univariate function: $g(x)$ and $h(n)$. The first way $g(x)$ is called an **iterate** of f in which n is considered constant, and the second way $h(n)$ is called an **orbit** of f in which x is considered constant.

The trend to use different univariate names for bivariate functions makes it much easier to talk about iteration, since it is easiest with univariate functions.

	$F(x,y) = f^y(x)$	$F(x,y) = x^y$	$F(x,y) = {}^y x$
$F(x,y)$	**iteration**	**exponentiation**	**tetration**
$G(x)$	(y-fold) **iterate**	(y-th) **power**	(y-th) **hyper-power**
$H(y)$	**orbit** (from x)	(base-x) **exponential**	(base-x) **tetrational**
x	state parameter	base	hyper-base
y	time parameter	exponent	hyper-exponent

Simply put, tetration (${}^y x$) is the y-fold iterate of the base-x exponential from 1.

3. Notations

3.1. Tetration Notation. Most authors use the *left-superscript* notation:

$$ {}^n x = x^{x^{\cdot^{\cdot^{x^x}}}} \quad \text{with } n\ x\text{'s} $$

which was invented by Hans Maurer in 1901. Since then the notation has been used by R. L. Goodstein, R. A. Knoebel, Rubtsov & Romerio, Rudy Rucker, Robert Munafo, Daniel Geisler, Eric W. Weisstein, Ioannis N. Galidakis, and many other authors. Many of these authors also refer to it as *tetration* (starting with Goodstein).

Notations for *iterated exponentials* vary from author to author, so the best way of writing iterated exponentials is with the iteration of $\exp_x(z) = x^z$:

$$ \exp_x^n(z) = x^{x^{\cdot^{\cdot^{x^z}}}} \quad \text{with } n\ x\text{'s} $$

Using this notation, tetration can be written ${}^n x = \exp_x^n(1)$.

3.2. Mixed Hyper-operator Notation. Mixed hyper-operator notation is based on two symbols: the *up-arrow* and the *down-arrow*. In the most general form, these can be used for any operator, but are usually used by themselves. These arrows take an operator (A) and make a new operator ($\uparrow A$) that is based on an iteration of the first operator. In computer science these two forms of evaluation are called **left-associative** and **right-associative** evaluation, but since we are talking about iteration we will use the terms left-associative and right-associative *iteration* instead.

(right) $$ x{\uparrow}Ay = xA(xA \cdots A(xAx)) \text{ with } y\ x\text{'s} $$

(left) $$ x{\downarrow}Ay = ((xAx)A \cdots Ax)Ax \text{ with } y\ x\text{'s} $$

One of the first things to notice is that if A is empty or missing, then $x{\uparrow}y$ or $x{\downarrow}y$ can both be interpreted as exponentiation x^y since they would both evaluate to y x's juxtaposed next to each other which (regardless of association) would be most naturally interpreted as multiplying y x's together, giving exponentiation.

Using this notation, tetration can be written ${}^y x = x{\uparrow\uparrow}y$.

3.3. **N-ary Tower Notation.** Nested exponentials and **towers** are synonymous in that they basically refer to the same thing. The Greek letters Σ and Π are used for n-ary sums and n-ary products respectively, so we will use T (Greek *tau*) for n-ary towers. This notation has been used by Barrow, Shell, Thron, Bachman, Brunson and Dong in their work on nested exponentials (although half of these authors used E (Greek *epsilon*) for exponential, while the other half used T for towers).

Tower notation can be found in many forms, but the two most useful are:

$$\mathop{\mathrm{T}}_{k=1}^{n} a_k = a_1^{a_2^{\cdot^{\cdot^{a_{n-1}^{a_n}}}}} \qquad \mathop{\mathrm{T}}_{k=1}^{n} (a_k; z) = a_1^{a_2^{\cdot^{\cdot^{a_n^{z}}}}}$$

Using this notation, tetration can be written ${}^{n}x = \mathrm{T}^n x$, and iterated exponentials can be written $\exp_x^n(z) = \mathrm{T}^n(x; z)$, assuming $(k = 1)$. The advantage of this notation for tetration and iterated exponentials is that there is no confusion with coefficients in front, i.e.: $c\mathrm{T}^n x$ is less confusing than $c\,{}^{n}x$, and it is more consistent than using both left-superscript and iteration notation for very similar things.

4. **Inverse Functions**

4.1. **Inverses of Tetration.** Just as there are two kinds of univariate functions based on tetration (hyper-powers and tetrationals), so are there two kinds of inverse functions of tetration. The inverses of hyper-power functions are called **super-roots** and the inverses of tetrational functions are called **super-logarithms** by analogy to the similar functions used with exponentiation. They are defined as follows:

$$\mathrm{srt}_n(z) = x \text{ if and only if } z = {}^{n}x$$
$$\mathrm{slog}_x(z) = n \text{ if and only if } z = {}^{n}x$$

4.2. **Infinitely Iterated Exponentials.** There are three functions that have many physical applications: the Lambert W-function $W(x)$, the second super-root $\mathrm{srt}_2(x)$, and infinitely iterated exponentials ${}^{\infty}x$. The Lambert W-function is defined as:

$$W(x) = w \text{ if and only if } w\boldsymbol{e}^w = x$$

which surprisingly, we can relate to the other two functions:

${}^{\infty}x$	$= \frac{W(-\ln x)}{-\ln x}$	$= \frac{1}{\mathrm{srt}_2(1/x)}$
${}^{\infty}(\boldsymbol{e}^{-x})x$	$= W(x)$	$= \frac{x}{\mathrm{srt}_2(\boldsymbol{e}^x)}$
$\frac{1}{{}^{\infty}(1/x)}$	$= \frac{\ln x}{W(\ln x)}$	$= \mathrm{srt}_2(x)$

Of all of these, the Lambert W-function is the most well-publicized for solving real-world problems, although any of these functions can be used for the same purpose.