Tables and Formulae of
Towers and Tetration

Eretrandre.org tetration forum members

1 Hyper-operations

Terminology. The prefix *hyper-* denotes all ranks, and the prefix *super-* denotes rank 4. The hyper-N-operation is also called hyper-N for short.

N	2	3	4
hyper-N-power	multiple	power	tower
hyper-N-exponential	multiple	exponential	tetrational
hyper-N-operation	multiplication	exponentiation	tetration
hyper-N-nest	n-ary product	nested exponential	nested tetrational
hyper-N-logarithm	division	logarithm	super-logarithm
hyper-N-root	division	root	super-root

If $N \in \mathbb{Z}$, then the above right-associative hyper-operation sequence is referred to. However, the case where N is a string of {L, R} symbols, refers to the *mixed* hyper-operators.

Notation. Using mnemonics, the table above can be written as $\text{hy}N\text{pow}^a(x)$, $\text{hy}N\exp_b(x)$, $\text{hyper}N(b,a)$, $\text{hy}N\text{nest}(a_1, a_2, \cdots, a_n)$, $\text{hy}N\log_b(x)$, and $\text{hy}N\text{rt}^a(x)$ respectively.

Using box notation, hyper-operations and their inverses can be written as:

$$x = b\boxed{N}a,\ \ b = {}^{a}\!\overset{N}{\lceil x} \text{ for hyper-roots, and } a = {}_{b}\underset{N}{\lfloor x} \text{ for hyper-logarithms.}$$

Arrow notation defines $b{\uparrow}Aa = bA(bA\cdots A(bAb))$ with a b's for right-iteration, and defines $b{\downarrow}Aa = ((bAb)A\cdots Ab)Ab$ with a b's for left-iteration. Arrow notation assigns both $\uparrow$ and $\downarrow$ to exponentiation, but some authors define these as tetration and hyper-L respectively. To avoid any confusion we explicitly use the symbol ($\hat{}$) for exponentiation.

N	Term	Notation	N	Term	Notation
L	hyper-L	$b{\downarrow}\hat{}\,a = b^{b^{(a-1)}}$	R	tetration	$b{\uparrow}\hat{}\,a = {}^{a}b$
LR	hyper-LR	$b{\downarrow}{\uparrow}\hat{}\,a$	RR	pentation	$b{\uparrow}{\uparrow}\hat{}\,a$
LL	hyper-LL	$b{\downarrow}{\downarrow}\hat{}\,a$	RL	hyper-RL	$b{\uparrow}{\downarrow}\hat{}\,a$

2 Scientific Notation and Large Numbers

Notation. Scientific notation represents a number in the form $n = c \times b^a = cb^a$ where a is the exponent, b is the base (usually $b = 10$), and c is the coefficient or mantissa. Tetrational notation represents a number in the form $n = \exp_b^a(c) = \exp_b^a \circ c = \exp_b^a c$ where a is the hyper-exponent, b is the base, and c is the argument. For example:

$$8080174247945128758864599049617107570057543680000000000 \approx \exp_{10}^2 1.7316485529$$

Both scientific notation an tetrational notation have standard cannonical forms[1]:

Term	Form	$b = 10$	a	c
scientific notation	$c \times b^a$	$c\mathrm{E}a$	$a \in \mathbb{Z}$	$1 \le c < b$
tetrational notation	$\exp_b^a(c)$	$c\mathrm{T}a$	$a \in \mathbb{Z}$	$1 \le c < b$

3 Powers and Exponentiation

Definitions. Exponentiation is a binary operation on $\mathbb{C}$ defined by:

$b^1 := b$	$b^a := bb^{a-1}$	$b^a := \boldsymbol{e}^{\ln(b)a}$	$\boldsymbol{e}^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$	$\ln(x) := \sum_{k=1}^{\infty} \frac{(1-x)^k}{-k}$

Terminology. A power function is a function of the form $x \mapsto x^n$, and an exponential function is a function of the form $x \mapsto b^x$. Exponentiation is the binary operation above.

Table of Values. Exponentiation near singularities can be summarized by:

	$a = -\infty$	$\Re(a) < 0$	$\Re(a) = 0$	$\Re(a) > 0$	$a = \infty$
$b = 0$	∞	∞	$\frac{0}{0}$	0	0
$0 < \lvert b \rvert < 1$	∞	b^a	b^a	b^a	0
$\lvert b \rvert = 1$	$\frac{0}{0}$	$\boldsymbol{e}^{i\theta}$	$\left\{ \begin{array}{ll} 1 & \text{if } a = 0, \\ 1 & \text{if } b = 1, \\ \boldsymbol{e}^{i\theta} & \text{otherwise} \end{array} \right\}$	$\boldsymbol{e}^{i\theta}$	$\frac{0}{0}$
$\lvert b \rvert > 1$	0	b^a	b^a	b^a	∞
$\lvert b \rvert = \infty$	0	0	$\frac{0}{0}$	∞	∞

where ∞ is complex infinity, and $\frac{0}{0}$ is an indeterminate value. The expression $\boldsymbol{e}^{i\theta}$ is meant to indicate that the result has magnitude 1, with the angle or argument: $\theta = a \operatorname{Arg}(b)$.

[1] The notation $c\mathrm{T}a$ has never been used, thus it is merely speculation at this point.

4 Nested Exponentials

Notation. Heterogeneous tower notation is defined by:

$$\mathop{\mathrm{T}}_{k=1}^{n} a_k := \mathop{\mathrm{T}}_{k=1}^{n-1} (a_k; a_n) := \mathrm{T}(a_1, a_2, \cdots, a_n) := a_1^{a_2^{\cdot^{\cdot^{\cdot^{a_{n-1}^{a_n}}}}}}$$

5 Towers and Tetration

Notation. Homogeneous tower notation is defined by:

$${}^{a}b := \mathrm{T}^{a}b := \mathop{\mathrm{T}}_{k=1}^{a} b = b\uparrow\hat{}\,a$$